AGRODEP Technical Note TN-02

November 2012

Functional Forms Commonly Used in CGE Models

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Abstract

This technical note presents a review of the functional forms commonly used to represent demand and production in Computable General Equilibrium (CGE) models. The general properties of demand and production functions are described. The note also includes the detailed derivation of some specific functions.

1 Demand functions

1.1 Generalities

Demand functions

Marshallian demand functions
Marshallian demand functions are derived from the maximization of direct utility $U(q_1, ..., q_n)$ under budget constraint:

$$\begin{cases} \max U(q_1, ..., q_n) \\ s.t. \sum_i p_i q_i = R \end{cases}$$

where q_i and p_i represent the demand and price respectively of good i, R is the household's income, and $U(q_1, ..., q_n)$ is the utility function. These demands are also called uncompensated, as opposed to the compensated Hicksian demand functions described bellow.

Marshallian demands are thus functions of prices and income: $q_i(p_1, p_2, ..., p_n, R)$.

From this we can find the indirect utility function $V(p_1, ..., p_n, R)$ which corresponds to the utility level obtained for a given income and set of prices.

$V(p_1, ..., p_n, R) = U(q_1, ..., q_n) = U(q_1(p_1, ..., p_n, R), ..., q_n(p_1, ..., p_n, R))$

Hicksian demand functions
Hicksian demand functions are derived from the minimization of expenditure under a utility constraint:

$$\begin{cases} \min \sum_i p_i q_i^H \\ s.t. U(q_1, ..., q_n) = u \end{cases}$$

Hicksian demands are also called compensated, since the utility level is held contant. They are thus functions of prices and utility: $q_i^H(p_1, p_2, ..., p_n, u)$.

From this we can find the expenditure function e which gives the minimum budget required to obtain a given utility level for a given set of prices.

$$e(p_1, ..., p_n, u) = \sum_i p_i q_i^H(p_1, ..., p_n, u)$$

It is also worth noting that:

$$q_i^H(p_1, p_2, ..., p_n, V(p_1, ..., p_n, R)) = q_i(p_1, p_2, ..., p_n, R)$$

In the same way:

$$q_i(p_1, p_2, ..., p_n, e(p_1, ..., p_n, u)) = q_i^H(p_1, p_2, ..., p_n, u)$$

Elasticities

Uncompensated price elasticities
The uncompensated price elasticities measure the evolution of demand with prices, the level of *income* being constant.

The own price elasticity measures the evolution of one good's demand with its own price:

$$\epsilon_i = \frac{\partial q_i}{\partial p_i}\frac{p_i}{q_i}$$

The cross price elasticity measures the evolution of one good's demand with the price of another good:

$$\epsilon_{ij} = \frac{\partial q_i}{\partial p_j}\frac{p_j}{q_i}$$

Income elasticities
The income elasticity measures the evolution of demand with income:

$$\eta_i = \frac{\partial q_i}{\partial R}\frac{R}{q_i}$$

Compensated price elasticities
The compensated (Hicksian) price elasticities measure the evolution of demand with prices, the level of *utility* being constant.

Own price elasticity:

$$\epsilon_i^H = \frac{\partial q_i^H}{\partial p_i}\frac{p_i}{q_i^H}$$

Cross price elasiticity:

$$\epsilon_{ij}^H = \frac{\partial q_i^H}{\partial p_j}\frac{p_j}{q_i^H}$$

The Allen substitution elasticity is defined as:

$$\sigma_{ij} = \frac{\epsilon_{ij}^H}{w_j}$$

Where w_j is the budget share of good j, that is, $w_j = \frac{p_j q_j}{\sum_i p_i q_i}$

Useful identities

Slutsky equation
The Slutsky equation decomposes the (uncompensated) price effect into two components: a substitution effect and an income effect. Namely, if the price of good j increases, the real income of the consumer decreases which induces a decrease in the demand for good i (income effect for normal goods), but as the price of good i relative to the price of good j decreases, the decrease in demand for good i may be attenuated (substitution effect for substituable goods). We thus have:

$$\epsilon_{ij} = \epsilon_{ij}^{H} - w_j \eta_i \Leftrightarrow \sigma_{ij} = \frac{\epsilon_{ij}}{w_j} + \eta_i$$

Shephard's lemma
The Shephard's lemma states that the demand for a particular good i for a given level of utility u and given prices p, equals the partial derivative of the expenditure function with respect to the price of the good:

$$q_i^H(p, u) = \frac{\partial e(p, u)}{\partial p_i}$$

This is easliy demonstrated by differentiating both sides of the equality:

$$e(p_1, ..., p_n, u) = \sum_i p_i q_i^H(p_1, ..., p_n, u)$$

Roy's identity
The Roy's identity relates the Marshallian demand function to the partial derivatives of the indirect utility function:

$$q_i(p, R) = -\frac{\frac{\partial V(p,R)}{\partial p_i}}{\frac{\partial V(p,R)}{\partial R}}$$

1.2 Cobb-Douglas

Demand function

The Cobb-Douglas demand function is derived from the following utility maximization program:

$$\begin{cases} \max U = \prod_j q_j^{\alpha_j} \\ s.t. \sum_j p_j q_j = R \end{cases}$$

With $\sum_i \alpha_i = 1$

$$\mathcal{L} = \prod_j q_j^{\alpha_j} - \lambda \left(\sum_j p_j q_j - R \right)$$

$$\frac{\partial \mathcal{L}}{\partial q_i} = 0 = \frac{\alpha_i \prod_j q_j^{\alpha_j}}{q_i} - \lambda p_i$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 = \sum_j p_j q_j - R$$

$$\lambda = \frac{\alpha_i \prod_j q_j^{\alpha_j}}{p_i q_i}, \forall i$$

$$\Rightarrow p_j q_j = \frac{\alpha_j p_i q_i}{\alpha_i}, \forall i, j$$

$$R = \sum_j p_j q_j = \frac{p_i q_i}{\alpha_i} \sum_j \alpha_j$$

$$\Rightarrow q_i = \frac{R\alpha_i}{p_i}$$

Elasticities

$$\epsilon_i = \frac{\partial q_i}{\partial p_i} \frac{p_i}{q_i} = -\frac{R\alpha_i}{p_i^2} \frac{p_i}{q_i} = -\frac{q_i}{p_i} \frac{p_i}{q_i} = -1$$

$$\epsilon_{ij} = \frac{\partial q_i}{\partial p_j} \frac{p_j}{q_i} = 0 \forall i \neq j$$

$$\eta_i = \frac{\partial q_i}{\partial R} \frac{R}{q_i} = \frac{\alpha_i}{p_i} \frac{R}{q_i} = \frac{\alpha_i}{p_i} \frac{Rp_i}{\alpha_i R} = 1$$

$$\sigma_{ij} = \frac{\epsilon_{ij}}{w_j} + \eta_i = 1$$

1.3 Constant Elasticity of Substitution

Demand function

The Constant Elasticity of Substitution (CES) demand function is derived from the following utility maximization program:

$$\begin{cases} \max U = \left(\sum_i \alpha_i {q_i}^{1-\frac{1}{\sigma}}\right)^{\frac{1}{1-\frac{1}{\sigma}}} \\ s.t. \sum_i p_i q_i = R \end{cases}$$

$$\mathcal{L} = \left(\sum_i \alpha_i {q_i}^{1-\frac{1}{\sigma}}\right)^{\frac{1}{1-\frac{1}{\sigma}}} - \lambda \left(\sum_i p_i q_i - R\right)$$

$$\frac{\partial \mathcal{L}}{\partial q_i} = 0 = \alpha_i {q_i}^{-\frac{1}{\sigma}} \left(\sum_j \alpha_j {q_j}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} - \lambda p_i$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 = \sum_j p_j q_j - R$$

$$\lambda = \frac{\alpha_i {q_i}^{-\frac{1}{\sigma}}}{p_i} \left(\sum_j \alpha_j {q_j}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}}, \forall i$$

$$\Rightarrow q_j = \left(\frac{\alpha_j p_i}{\alpha_i p_j}\right)^{\sigma} q_i, \forall i, j$$

$$R = \sum_j p_j q_j = q_i \left(\frac{p_i}{\alpha_i}\right)^{\sigma} \sum_j \alpha_j^{\sigma} p_j^{1-\sigma}$$

$$\Rightarrow q_i = \frac{\alpha_i^{\sigma} R}{p_i^{\sigma} \sum_j \alpha_j^{\sigma} p_j^{1-\sigma}}$$

Elasticities

$$\epsilon_i = \frac{\partial q_i}{\partial p_i}\frac{p_i}{q_i} = \frac{p_i}{q_i}\left(\frac{-\alpha_i^\sigma R\sigma p_i^{\sigma-1}\sum_j \alpha_j^\sigma p_j^{1-\sigma} - \alpha_i^{2\sigma} R\left(1-\sigma\right)}{\left(p_i^\sigma \sum_j \alpha_j^\sigma p_j^{1-\sigma}\right)^2}\right) = -\sigma + \frac{\alpha_i^\sigma\left(\sigma-1\right)}{p_i^{\sigma-1}\sum_j \alpha_j^\sigma p_j^{1-\sigma}}$$

$$\epsilon_{ij} = \frac{\partial q_i}{\partial p_j}\frac{p_j}{q_i} = \left(\sigma-1\right)\frac{p_j q_j}{R}, \forall i \neq j$$

$$\eta_i = \frac{\partial q_i}{\partial R}\frac{R}{q_i} = \frac{\alpha_i^\sigma}{p_i^\sigma \sum_j \alpha_j^\sigma p_j^{1-\sigma}}\frac{R}{q_i} = 1$$

$$\sigma_{ij} = \frac{\epsilon_{ij}}{w_j} + \eta_i = \left(\sigma-1\right) + 1 = \sigma$$

Where the second equation follows from noting that $w_j = \frac{p_j q_j}{R} = \frac{p_j}{R}\frac{\alpha_j R}{p_j^\sigma \sum_k \alpha_k p_k^{1-\sigma}} = \frac{\alpha_j p_j^{1-\sigma}}{\sum_k \alpha_k p_k^{1-\sigma}}, \forall j$.

1.4 Linear Expenditure System-Cobb Douglas

Demand function

The Linear Expenditure System (LES)- Cobb-Douglas demand function is derived from the Stone-Geary utility maximization program:

$$\begin{cases} \max U = \prod_i \left(q_i - qmin_i\right)^{\alpha_i} \\ s.t. \sum_i p_i q_i = R \end{cases}$$

With $\sum_i \alpha_i = 1$; $qmin_i$ being the minimal consumption quantities.

$$\mathcal{L} = \prod_i \left(q_i - qmin_i\right)^{\alpha_i} - \lambda\left(\sum_i p_i q_i - R\right)$$

$$\frac{\partial \mathcal{L}}{\partial q_i} = 0 = \frac{\alpha_i}{q_i - qmin_i}\prod_j \left(q_j - qmin_j\right)^{\alpha_j} - \lambda p_i$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 = \sum_j p_j q_j - R$$

$$\lambda = \frac{\alpha_i}{p_i\left(q_i - qmin_i\right)} \prod_j \left(q_j - qmin_j\right)^{\alpha_j}, \forall i$$

$$\Rightarrow q_j = qmin_j + \frac{\alpha_j p_i}{\alpha_i p_j}\left(q_i - qmin_i\right), \forall i, j$$

$$R = \sum_j p_j q_j = \sum_j p_j qmin_j + \left(q_i - qmin_i\right)\frac{p_i}{\alpha_i}\sum_j \alpha_j$$

$$\Rightarrow q_i = qmin_i + \frac{\alpha_i}{p_i}\left(R - \sum_j p_j qmin_j\right)$$

Elasticities

$$\epsilon_i = \frac{\partial q_i}{\partial p_i}\frac{p_i}{q_i} = -\frac{p_i}{q_i}\left(\frac{\alpha_i}{p_i^2}\left(R - \sum_j p_j qmin_j\right) + \frac{\alpha_i}{p_i} qmin_i\right) = \frac{qmin_i}{q_i}\left(1 - \alpha_i\right) - 1$$

$$\epsilon_{ij} = \frac{\partial q_i}{\partial p_j}\frac{p_j}{q_i} = -\frac{\alpha_i p_j qmin_j}{p_i q_i}, \forall i \neq j$$

$$\eta_i = \frac{\partial q_i}{\partial R}\frac{R}{q_i} = \alpha_i \frac{R}{p_i q_i}$$

$$\sigma_{ij} = \frac{\epsilon_{ij}}{w_j} + \eta_i = \frac{\alpha_i}{w_i}\left(1 - \frac{qmin_j}{q_j}\right)$$

1.5 Linear Expenditure System - Constant Elasticity of Substitution

Demand function

The LES-CES demand function is derived from the following utility maximization program:

$$\begin{cases} \max U = \left(\sum_i \alpha_i^{\frac{1}{\sigma}}\left(q_i - qmin_i\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \\ s.t. \sum_i p_i q_i = R \end{cases}$$

$$\mathcal{L} = \left(\sum_i \alpha_i^{\frac{1}{\sigma}} \left(q_i - qmin_i \right)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} - \lambda \left(\sum_i p_i q_i - R \right)$$

$$\frac{\partial \mathcal{L}}{\partial q_i} = 0 = \alpha_i^{\frac{1}{\sigma}} \left(q_i - qmin_i \right)^{-\frac{1}{\sigma}} \left(\sum_j \alpha_j^{\frac{1}{\sigma}} \left(q_j - qmin_j \right)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{1}{\sigma-1}} - \lambda p_i$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 = \sum_j p_j q_j - R$$

$$\lambda = \frac{\alpha_i^{\frac{1}{\sigma}} \left(q_i - qmin_i \right)^{-\frac{1}{\sigma}}}{p_i} \left(\sum_j \alpha_j^{\frac{1}{\sigma}} \left(q_j - qmin_j \right)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{1}{\sigma-1}}, \forall i$$

$$\Rightarrow q_j = qmin_j + \frac{\alpha_j p_i^{\sigma}}{\alpha_i p_j^{\sigma}} \left(q_i - qmin_i \right), \forall i, j$$

$$R = \sum_j p_j q_j = \sum_j p_j qmin_j + \left(q_i - qmin_i \right) \frac{p_i^{\sigma}}{\alpha_i} \sum_j \frac{\alpha_j}{p_j^{\sigma-1}}$$

$$\Rightarrow q_i = qmin_i + \frac{\alpha_i \left(R - \sum_j p_j qmin_j \right)}{p_i^{\sigma} \sum_j \alpha_j p_j^{1-\sigma}}$$

Elasticities

$$\epsilon_i = -\frac{\alpha_i p_i^{1-\sigma}}{q_i \sum_j \alpha_j p_j^{1-\sigma}} \left(q_i + \sigma \left(\frac{\left(R - \sum_j p_j qmin_j \right)}{p_i} - q_i + qmin_i \right) \right)$$

$$\epsilon_{ij} = -\frac{\alpha_i p_j \left(q_j - \sigma \left(q_j - qmin_j \right) \right)}{q_i p_i^{\sigma} \sum_k \alpha_k p_k^{1-\sigma}}, \forall i, j$$

$$\eta_i = \frac{\partial q_i}{\partial R} \frac{R}{q_i} = \frac{R \left(q_i - qmin_i \right)}{q_i \left(R - \sum_j p_j qmin_j \right)}$$

$$\sigma_{ij} = \frac{\epsilon_{ij}}{w_j} + \eta_i = \frac{\sigma R \left(q_i - qmin_i \right) \left(q_j - qmin_j \right)}{q_i q_j \left(R - \sum_k p_k qmin_k \right)}$$

1.6 Almost Ideal Demand System

Demand function

The Almost Ideal Demand System (AIDS), proposed by Deaton and Muellbauer (1980) is a flexible functional is the sense that it has enough parameters to be regarded as a reasonable approximation to whatever the true unknown function might be.

The consumers' preferences on which the AIDS is based, are assumed to belong to the Price Independant Generalized linear Log (PIGLOG) class of preferences and are represented *via* the expenditure function:

$$\ln\left(e\left(p,u\right)\right)=\alpha_0+\sum_k \alpha_k \ln p_k+\frac{1}{2}\sum_k\sum_j \gamma*_{kj}\ln p_k \ln p_j+u\beta_0\prod_k p_k^{\beta_k} \quad (1)$$

Where $\sum_i \alpha_i=1$, $\sum_j \gamma*_{kj}=\sum_j \gamma*_{jk}=\sum_j \beta_j=0$

As $e\left(p,u\right)=\sum_i p_i q_i$, we know that $\frac{\partial e(p,u)}{\partial p_i}=q_i$, which leads to:

$$\frac{p_i \partial e\left(p,u\right)}{e\left(p,u\right)\partial p_i}=\frac{p_i q_i}{e\left(p,u\right)} \quad (2)$$

Recognizing that $w_i=\frac{p_i q_i}{e(p,u)}$ and that $\frac{p_i\partial(p,u)}{e(p,u)\partial p_i}=\frac{\partial\ln(e(p,u))}{\partial \ln p_i}$, using equality (2) and logarithmic differentiation of (1) leads to:

$$w_i=\alpha_i+\sum_j \gamma_{ij}\ln p_j+\beta_i u\beta_0\prod_k p_k^{\beta_k}$$

With $\gamma_{ij}=\frac{1}{2}\left(\gamma*_{ij}+\gamma*_{ji}\right)$

Furthermore, (1) implies that:

$$\ln\left(e\left(p,u\right)\right)=\alpha_0+\sum_k \alpha_k \ln p_k+\frac{1}{2}\sum_k\sum_j \gamma*_{kj}\ln p_k \ln p_j+u\beta_0\prod_k p_k^{\beta_k}$$

$$\Rightarrow u\beta_0\prod_k p_k^{\beta_k}=\ln\left(e\left(p,u\right)\right)-\left(\alpha_0+\sum_k \alpha_k \ln p_k+\frac{1}{2}\sum_k\sum_j \gamma*_{kj}\ln p_k \ln p_j\right)$$

The AIDS is thus expressed as:

$$w_i=\frac{p_i q_i}{R}=\alpha_i+\sum_j \gamma_{ij}\ln p_j+\beta_i \ln\left(\frac{R}{P}\right)$$

With $\gamma_{ij}=\gamma_{ji}$, $\sum_i \alpha_i=1$, $\sum_i \beta_i=0$ and $\sum_j \gamma_{ij}=0$

And P a price index such that: $\ln P=\alpha_0+\sum_k \alpha_k \ln p_k+\frac{1}{2}\sum_k\sum_j \gamma_{kj}\ln p_k \ln p_j$

Elasticities

$$\epsilon_i = \frac{\partial q_i}{\partial p_i}\frac{p_i}{q_i} = -1 + \frac{\gamma_{ii}}{w_i} - \frac{\beta_i}{w_i}\left(\alpha_i + \sum_j \gamma_{ji}\ln p_j\right)$$

$$\epsilon_{ij} = \frac{\gamma_{ij}}{w_i} - \frac{\beta_i}{w_i}\left(\alpha_j + \sum_k \gamma_{kj}\ln p_k\right)$$

$$\eta_i = \frac{\partial q_i}{\partial R}\frac{R}{q_i} = 1 + \frac{\beta_i}{w_i}$$

$$\sigma_{ij} = \frac{\epsilon_{ij}}{w_j} + \eta_i = 1 + \frac{\gamma_{ij}}{w_i w_j} + \frac{\beta_i\beta_j}{w_i w_j}\ln\left(\frac{R}{P}\right)$$

1.7 Linearized Almost Ideal Demand System

Demand function

In this linearized version of the AIDS, the non linear price index P is replaced by the Stone's price index P^* to ease the empirical use of the model. The equation definig this index P^* is given by:

$$\ln P^* = \sum_k w_k \ln p_k$$

The demand is thus equal to:

$$q_i = \frac{R}{p_i}\left(\alpha_i + \sum_j \gamma_{ij}\ln p_j + \beta_i\left(\ln R - \sum_j \frac{p_j q_j}{R}\ln p_j\right)\right)$$

$$\Rightarrow q_i = \frac{R}{p_i}\left(\alpha_i + \sum_j \gamma_{ij}\ln p_j + \beta_i\left(\ln R - \sum_{j\neq i} \frac{p_j q_j}{R}\ln p_j - \frac{p_i q_i}{R}\ln p_i\right)\right)$$

$$\Rightarrow q_i = \frac{R}{(1+\beta_i\ln p_i)\,p_i}\left(\alpha_i + \sum_j \gamma_{ij}\ln p_j + \beta_i\left(\ln R - \sum_{j\neq i} \frac{p_j q_j}{R}\ln p_j\right)\right)$$

Elasticities

$$\epsilon_i = \frac{\partial q_i}{\partial p_i}\frac{p_i}{q_i} = -\frac{1+\beta_i \ln p_i + \beta_i}{1+\beta_i \ln p_i} + \frac{1}{1+\beta_i \ln p_i}\left(\frac{\gamma_{ii}}{w_i} - \frac{\beta_i}{w_i}\left(\sum_{i\neq j}\frac{\partial q_j p_i}{\partial p_i q_j}\frac{q_j p_j}{R}\ln p_j\right)\right)$$

$$\Rightarrow \epsilon_i = -1 + \frac{1}{1+\beta_i \ln p_i}\left(-\beta_i + \frac{\gamma_{ii}}{w_i} + \frac{\beta_i}{w_i}\left(\sum_{j\neq i}\epsilon_{ji} w_j \ln p_j\right)\right)$$

$$\Rightarrow \epsilon_i = -1 - \beta_i + \frac{\gamma_{ii}}{w_i} - \frac{\beta_i}{w_i}\left(w_i \ln p_i + \sum_j \epsilon_{ji} w_j \ln p_j\right)$$

$$\epsilon_{ij} = \frac{\partial q_i}{\partial p_j}\frac{p_j}{q_i} = \frac{1}{w_i\left(1+\beta_i \ln p_i\right)}\left(\gamma_{ij} - \beta_i\left(w_j \ln p_j + w_j + \sum_{k\neq i}\epsilon_{kj} w_k \ln p_k\right)\right)$$

$$\Rightarrow \epsilon_{ij} = \frac{w_j \beta_i \ln p_j}{w_i\left(1+\beta_i \ln p_i\right)} + \frac{1}{1+\beta_i \ln p_i}\left(\frac{\gamma_{ij}}{w_i} - \frac{\beta_i w_j}{w_i} + \beta_i \ln p_i \epsilon_{ij} - \frac{\beta_i}{w_i}\sum_k w_k \ln p_k \epsilon_{kj}\right)$$

$$\Rightarrow \epsilon_{ij} = \frac{\gamma_{ij}}{w_i} - \frac{\beta_i w_j}{w_i} - \frac{\beta_i}{w_i}\left(w_j \ln p_j + \sum_k w_k \ln p_k \epsilon_{kj}\right)$$

$$\eta_i = \frac{\partial q_i}{\partial R}\frac{R}{q_i} = 1 + \frac{\beta_i}{w_i\left(1+\beta_i \ln p_i\right)}\left(1 - \sum_{i\neq j} w_j \ln p_j\left(\eta_j - 1\right)\right)$$

$$\Rightarrow \left(\eta_i - 1\right)\left(1+\beta_i \ln p_i\right) w_i = \beta_i + \beta_i w_i \ln p_i\left(\eta_i - 1\right) - \beta_i \sum_j w_j \ln p_j\left(\eta_j - 1\right)$$

$$\Rightarrow \eta_i = 1 + \frac{\beta_i}{w_i}\left(1 + \ln P^* - \sum_j \eta_j w_j \ln p_j\right)$$

$$\sigma_{ij} = \frac{\epsilon_{ij}}{w_j} + \eta_i = 1 + \frac{\gamma_{ij}}{w_j w_i} - \frac{\beta_i}{w_j w_i}\left(w_j \ln p_j - w_j \ln P^* + \sum_k\left(w_k \ln p_k \epsilon_{kj} + w_j \eta_k w_k \ln p_k\right)\right)$$

$$\Rightarrow \sigma_{ij} = 1 + \frac{\gamma_{ij}}{w_j w_i} - \frac{\beta_i}{w_i}\left(\ln\frac{p_j}{P^*} + \sum_k \sigma_{kj} w_k \ln p_k\right)$$

1.8 Normalized Quadratic Expenditure System

Demand function

We start from the normalized quadratic expenditure function:

$$e(p,u) = \sum_j a_j p_j + \left(\sum_j b_j p_j + \frac{1}{2}\frac{\sum_j \sum_k \beta_{jk} p_j p_k}{\sum_j \alpha_j p_j}\right) u$$

With $\beta_{ij} = \beta_{ji}$

As $e(p,u) = R$, the indirect utility function can be expressed as:

$$V(p,R) = \frac{\left(R - \sum_j a_j p_j\right)}{\left(\sum_j b_j p_j + \frac{1}{2}\frac{\sum_j \sum_k \beta_{jk} p_j p_k}{\sum_j \alpha_j p_j}\right)}$$

Furthermore, applying Shephard's lemma, allows us to find the Hicksian demand function:

$$q_i^H(p,u) = \frac{\partial e(p,u)}{\partial p_i} = a_i + \left(b_i + \frac{\sum_j p_j \beta_{ij}}{\sum_j \alpha_j p_j} - \frac{1}{2}\frac{\alpha_i \sum_j \sum_k \beta_{jk} p_j p_k}{\left(\sum_j \alpha_j p_j\right)^2}\right) u$$

Replacing u with the expression for $V(p,R)$ in this equation, we find the Marsallian demand function:

$$q_i(p,R) = a_i + \frac{\left(R - \sum_j a_j p_j\right)\left(b_i + \frac{\sum_j p_j \beta_{ij}}{\sum_j \alpha_j p_j} - \frac{1}{2}\frac{\alpha_i \sum_j \sum_k \beta_{jk} p_j p_k}{\left(\sum_j \alpha_j p_j\right)^2}\right)}{\left(\sum_j b_j p_j + \frac{1}{2}\frac{\sum_j \sum_k \beta_{jk} p_j p_k}{\sum_j \alpha_j p_j}\right)}$$

Elasticities

$$\epsilon_i = \frac{\partial q_i}{\partial p_i}\frac{p_i}{q_i} = \frac{p_i}{q_i}\left(-\frac{a_i\left(b_i + \frac{\sum_j \beta_{ij} p_j}{\sum_j \alpha_j p_j} - \frac{1}{2}\frac{\alpha_i \sum_j \sum_k \beta_{jk} p_j p_k}{\left(\sum_j \alpha_j p_j\right)^2}\right)}{\sum_j b_j p_j + \frac{1}{2}\frac{\sum_j \sum_k \beta_{jk} p_j p_k}{\sum_j \alpha_j p_j}} \right.$$
$$+ \frac{\left(R - \sum_j a_j p_j\right)\left(\frac{\beta_{ii}}{\sum_j \alpha_j p_j} - 2\frac{\alpha_i \sum_j \beta_{ij} p_j}{\left(\sum_j \alpha_j p_j\right)^2} + \frac{\alpha_i^2 \sum_j \sum_k \beta_{jk} p_j p_k}{\left(\sum_j \alpha_j p_j\right)^3}\right)}{\sum_j b_j p_j + \frac{1}{2}\frac{\sum_j \sum_k \beta_{jk} p_j p_k}{\sum_j \alpha_j p_j}}$$
$$\left. - \frac{(q_i - a_i)^2}{R - \sum_j \alpha_j p_j}\right)$$

$$\Rightarrow \epsilon_i = -\frac{p_i\left(q_i - a_i\right)}{R - \sum_j a_j p_j} + \frac{p_i\left(R - \sum_j a_j p_j\right)\left(\beta_{ii} - 2\frac{\alpha_i \sum_j \beta_{ij} p_j}{\sum_j \alpha_j p_j} + \frac{\alpha_i^2 \sum_j \sum_k \beta_{jk} p_j p_k}{\left(\sum_j \alpha_j p_j\right)^2}\right)}{q_i\left(\sum_j \alpha_j p_j \sum_j b_j p_j + \frac{1}{2}\sum_j \sum_k \beta_{jk} p_j p_k\right)}$$

$$\epsilon_{ij} = \frac{\partial q_i}{\partial p_j}\frac{p_j}{q_i} = -\frac{p_j q_j\left(q_i - a_i\right)}{q_i\left(R - \sum_k a_k p_k\right)} + \frac{p_j\left(R - \sum_k a_k p_k\right)\left(\beta_{ij} - \frac{\sum_k\left(\alpha_j \beta_{ik} + \alpha_i \beta_{jk}\right)p_k}{\sum_k \alpha_k p_k} + \frac{\alpha_i \alpha_j \sum_k \sum_l \beta_{kl} p_k p_l}{\left(\sum_k \alpha_k p_k\right)^2}\right)}{q_i\left(\sum_k \alpha_k p_k \sum_k b_k p_k + \frac{1}{2}\sum_k \sum_l \beta_{kl} p_k p_l\right)}$$

$$\eta_i = \frac{\partial q_i}{\partial R}\frac{R}{q_i} = \frac{R\left(b_i + \frac{\sum_j \beta_{ij} p_j}{\sum_j \alpha_j p_j} - \frac{1}{2}\frac{\alpha_i \sum_j \sum_k \beta_{jk} p_j p_k}{\left(\sum_j \alpha_j p_j\right)}\right)}{q_i\left(\sum_j b_j p_j + \frac{1}{2}\frac{\sum_j \sum_k \beta_{jk} p_j p_k}{\sum_j \alpha_j p_j}\right)}$$

$$\Rightarrow \eta_i = \frac{R\left(q_i - a_i\right)}{q_i\left(R - \sum_j a_j p_j\right)}$$

$$\sigma_{ij} = \frac{\epsilon_{ij}}{w_j} + \eta_i = \frac{R\left(R - \sum_k a_k p_k\right)\left(\beta_{ij} - \frac{\sum_k\left(\alpha_j \beta_{ik} + \alpha_i \beta_{jk}\right)p_k}{\sum_k \alpha_k p_k} + \frac{\alpha_i \alpha_j \sum_k \sum_l \beta_{kl} p_k p_l}{\left(\sum_k \alpha_k p_k\right)^2}\right)}{q_i q_j\left(\sum_k \alpha_k p_k \sum_k b_k p_k + \frac{1}{2}\sum_k \sum_l \beta_{kl} p_k p_l\right)}$$

Normalization

A normalization of the NQES parameters is needed in order to avoid indeterminancy problems in their estimation (see Diewert and Fox, 2009). Namely, for a set of reference prices ${p^*}_i$:

$$\sum_i b_i {p^*}_i = 1$$

$$\sum_j \beta_{ij} {p^*}_j = 0$$

$$\sum_i a_i {p^*}_i = 0$$

If, as usual in CGE modelling, the reference prices are set to 1, which eases the calibration of parameters.

Indeed, in that case, at initial point we have:

$$q_i = a_i + Rb_i$$

$$\epsilon_i = -b_i + \frac{R\beta_{ii}}{q_i \sum_j \alpha_j}$$

$$\epsilon_{ij} = -\frac{q_j \left(q_i - a_i\right)}{q_i R} + \frac{R\beta_{ij}}{q_i \sum_k \alpha_k}$$

$$\eta_i = \frac{Rb_i}{q_i}$$

$$\sigma_{ij} = \frac{R^2 \beta_{ij}}{q_i q_j \sum_k \alpha_k}$$

1.9 An Implicit Direct Additive Demand System

Demand function

Here we start from an implicit directly additive utility function (see Hanoch (1975) for more detail on implicit additive utility functions):

$$\sum_i \frac{\alpha_i + \beta_i e^u}{1 + e^u} ln \left(\frac{q_i - qmin_i}{Ae^u} \right) = 1$$

Where $qmin_i$ represents the minimal possible consumption level of good i, and A, α_i, β_i are parameters, with $\alpha_i \geq 0$, $\sum_i \alpha_i = 1$, $\beta_i \leq 1$ and $\sum_i \beta_i = 1$.

The AIDADS (An Implicit Direct Additive Demand System) is derived from the maximization of utility u, subject to its implicit additivity and to a budget constraint:

$$\begin{cases} \max u \\ s.t. \sum_i \frac{\alpha_i+\beta_i e^u}{1+e^u} ln \frac{q_i - qmin_i}{Ae^u} = 1 \\ s.t. \sum_{p_i q_i} = R \end{cases}$$

$$\mathcal{L} = u - \lambda \left(\sum_i \frac{\alpha_i + \beta_i e^u}{1+e^u} ln \frac{q_i - qmin_i}{Ae^u} - 1 \right) - \xi \left(\sum_{p_i q_i} -R \right)$$

Let's define $\phi_i = \frac{\alpha_i+\beta_i e^u}{1+e^u}$

$$\frac{\partial \mathcal{L}}{\partial q_i} = 0 = \frac{\partial u}{\partial q_i} - \lambda \left(\frac{\partial u}{\partial q_i} \sum_j \left(\frac{\partial \phi_j}{\partial u} \ln \left(\frac{q_j - qmin_j}{Ae^u} \right) - phi_j \right) + \frac{\phi_i}{q_i - qmin_i} \right) + \xi p_i$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 = 1 - \sum_j \phi_j \ln \left(\frac{q_j - qmin_j}{Ae^u} \right) \Rightarrow \sum_j \phi_j \ln \left(\frac{q_j - qmin_j}{Ae^u} \right) = 1$$

$$\Rightarrow \frac{\partial u}{\partial q_i} \sum_j \left(\frac{\partial \phi_j}{\partial u} \ln \left(\frac{q_j - qmin_j}{Ae^u} \right) - \phi_j \right) + \frac{\phi_i}{(q_i - qmin_i)} = 0$$

We thus have:

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 = \frac{\partial u}{\partial q_i} - \xi p_i \Rightarrow \frac{\partial u}{\partial q_i} = \xi p_i$$

And:

$$\frac{\partial u}{\partial q_i} = - \frac{\phi_i}{(q_i - qmin_i) \sum_j \left(\frac{\partial \phi_j}{\partial u} \ln \left(\frac{q_j - qmin_j}{Ae^u} \right) - \phi_j \right)}$$

From the two previous equalities we have:

$$\xi = - \frac{\phi_i}{p_i (q_i - qmin_i) \sum_j \left(\frac{\partial \phi_j}{\partial u} \ln \left(\frac{q_j - qmin_j}{Ae^u} \right) - \phi_j \right)}, \forall i$$

$$\Rightarrow q_j = qmin_j + \frac{\phi_j p_i (q_i - qmin_i)}{\phi_i p_j}, \forall i, j$$

$$\frac{\partial \mathcal{L}}{\partial \xi} = 0 = \sum_j p_j q_j - R$$

$$\Rightarrow R = \sum_j p_j qmin_j + \frac{p_i (q_i - qmin_i)}{\phi_i}$$

$$\Rightarrow q_i = qmin_i + \frac{\phi_i \left(R - \sum_j p_j qmin_j\right)}{p_i} = qmin_i + \frac{(\alpha_i + \beta_i e^u) \left(R - \sum_j p_j qmin_j\right)}{p_i (1 + e^u)}$$

We can also derive the the Hicksian AIDADS (An Implicit Direct Additive Demand System) from the minimization of expenditure, subject the utility constraint:

$$\begin{cases} \min \sum_{p_i q_i} = R \\ s.t. \sum_i \frac{\alpha_i + \beta_i e^u}{1+e^u} \ln \frac{q_i - qmin_i}{Ae^u} = 1 \end{cases}$$

$$\begin{aligned} \mathcal{L} &= \textstyle\sum_i p_i q_i + \lambda \left(\sum_i \frac{(\alpha_i + \beta_i e^u)}{(1+e^u)} \ln \left(\frac{q_i - qmin_i}{Ae^u} \right) - 1 \right) \\ &= \textstyle\sum_i p_i q_i + \lambda \left(\sum_i \phi_i \ln \left(\frac{q_i - qmin_i}{Ae^u} \right) - 1 \right) \end{aligned}$$

$$\frac{\partial \mathcal{L}}{\partial q_i} = 0 = p_i + \lambda \left(\frac{\phi_i}{q_i - \gamma_i} \right)$$

$$\Rightarrow q_j = \frac{p_i \phi_j (q_i - qmin_i)}{p_j \phi_i} + qmin_j, \forall i, j$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 = \sum_j \phi_j \ln \left(\frac{q_j - qmin_j}{Ae^u} \right) - 1$$

$$\Rightarrow \sum_j \phi_j \ln \left(\frac{\phi_j}{p_j} \right) + \ln \left(\frac{p_i}{\phi_i A e^u} \right) = 1 - \ln (q_i - qmin_i)$$

$$\Rightarrow q^H{}_i = qmin_i + Ae^{u+1} \left(\frac{\phi_i}{p_i} \right) \prod_j \left(\frac{p_j}{\phi_j} \right)^{\phi_j}$$

Elasticities

$$\eta_i = \frac{\partial q_i}{\partial R}\frac{R}{q_i} = \frac{R}{q_i}\frac{1}{p_i}\left(\frac{\partial \phi_i}{\partial R}\left(R - \sum_j p_j qmin_j\right) + \phi_i\right)$$

Yet

$$\frac{\partial \phi_i}{\partial R} = \frac{\partial \phi_i}{\partial u}\left(\sum_j \frac{\partial u}{\partial q_j}\frac{\partial q_j}{\partial R}\right)$$

As $\phi_i = \frac{\alpha_i + \beta_i e^u}{1+e^u}$, we have:

$$\frac{\partial \phi_i}{\partial u} = \frac{(\beta_i - \alpha_i)\, e^u}{(1+e^u)^2}$$

So:

$$\eta_i = \frac{R}{q_i}\frac{1}{p_i}\left(\left(R - \sum_j p_j qmin_j \frac{(\beta_i - \alpha_i)\, e^u}{(1+e^u)^2}\left(\sum_j \frac{\partial u}{\partial q_j}\frac{\partial q_j}{\partial R}\right)\right) + \phi_i\right)$$

And, as we have seen:

$$\frac{\partial u}{\partial q_i} = -\frac{\phi_i}{(q_i - qmin_i)\sum_j\left(\frac{\partial \phi_j}{\partial u}\ln\left(\frac{q_j - qmin_j}{Ae^u}\right) - \phi_j\right)}$$

$$\Rightarrow \frac{\partial u}{\partial q_i} = -\frac{p_i}{\left(R - \sum_j p_j qmin_j\right)\left(\frac{e^u}{(1+e^u)^2}\sum_j\left((\beta_j - \alpha_j)\ln(q_j - qmin_j)\right) - 1\right)}$$

So, we have:

$$\eta_i = \frac{R}{q_i p_i}\left(\phi_i - \left(\frac{\left(R - \sum_j p_j qmin_j\right)\frac{(\beta_i - \alpha_i)e^u}{(1+e^u)^2}\sum_j p_j \frac{\partial q_j}{\partial R}}{\left(R - \sum_j p_j qmin_j\right)\left(\frac{e^u}{(1+e^u)^2}\sum_j\left((\beta_j - \alpha_j)\ln(q_j - qmin_j)\right) - 1\right)}\right)\right)$$

Furthermore, as $R = \sum_j p_j q_j$, $\sum_j p_j \frac{\partial q_j}{\partial R} = 1$
Then:

$$\eta_i = \frac{1}{w_i}\left(\phi_i - \left(\frac{\beta_i - \alpha_i}{\sum_j \left((\beta_j - \alpha_j)\ln\left(q_j - qmin_j\right)\right) - \frac{(1+e^u)^2}{e^u}}\right)\right)$$

We also have:

$$\epsilon^H{}_i = \frac{\partial q^H{}_i}{\partial p_i}\frac{p_i}{q^H{}_i} = \frac{p_i}{q^H{}_i}\left(-\frac{1}{p_i}Ae^{u+1}\left(\frac{\phi_i}{p_i}\right)\prod_j\left(\frac{p_j}{\phi_j}\right)^{\phi_j} + \frac{\phi_i}{p_i}Ae^{u+1}\left(\frac{\phi_i}{p_i}\right)\right)\prod_j\left(\frac{p_j}{\phi_j}\right)^{\phi_j}$$

$$\Rightarrow \epsilon^H{}_i = (\phi_i - 1)\frac{\left(q^H{}_i - qmin_i\right)}{q^H{}_i} = (\phi_i - 1)\frac{p_i\left(q^H{}_i - qmin_i\right)}{w_i R}$$

$$\epsilon^H{}_{ij} = \frac{\partial q^H{}_i}{\partial p_j}\frac{p_j}{q^H{}_i} = \frac{p_j}{q^H{}_i}\left(\frac{\phi_i\phi_j}{p_i}\right)Ae^{u+1}\frac{\phi_j}{p_j}\prod_k\left(\frac{p_k}{\phi_k}\right)^{\phi_k}, \forall i \neq j$$

$$\Rightarrow \epsilon^H{}_{ij} = \frac{\phi_i p_j\left(q_j - qmin_j\right)}{w_i R}, \forall i \neq j$$

Therefore:

$$\sigma_{ij} = \frac{\epsilon^H{}_{ij}}{w_j} = \frac{\phi_i}{w_i R}\left(\frac{p_j\left(q_j - qmin_j\right)}{w_j}\right)$$

And from Slutsky's equation,

$$\epsilon_{ij} = \sigma_{ij}w_j - \eta_i w_j$$

$$\Rightarrow \epsilon_{ij} = -\frac{\phi_i p_j qmin_j}{w_i R} + \frac{w_j}{w_i}\left(\frac{(\beta_i - \alpha_i)}{\left(\sum_j \left((\beta_j - \alpha_j)\ln\left(q_j - qmin_j\right)\right) - \frac{(1+e^u)^2}{e^u}\right)}\right)$$

1.10 Degrees of freedom

Demand system	Parameters	Restrictions on parameters	Degrees of freedom
Cobb-Douglas	α_i	$\sum_i \alpha_i = 1$	$n-1$
LES-CES	α_i, σ $qmin_i$	$\sum_i {\alpha_i}^{\frac{1}{\sigma}} = 1$	$2n$
AIDS	α_0, α_i, β_i, γ_{ij}	$\sum_i \alpha_i = 1$, $\sum_i \beta_i = 0$ $\sum_i \gamma_{ij} = 0$, $\gamma_{ij} = \gamma_{ji}$	$2n-1$ $+\frac{n(n-1)}{2}$
NQES	a_i, α_i b_i, β_{ij}	$\sum_i b_i {p^*}_i = 1$, $\sum_i a_i {p^*}_i = 0$ $\sum_j \beta_{ij} {p^*}_j = 0$, $\beta_{ij} = \beta_{ji}$ $\alpha_i = initialshare$	$3n-2$ $+\frac{n(n-1)}{2}$
AIDADS	α_i, β_i σ, $qmin_i$	$\sum_i \alpha_i = 1$, $\sum_i \beta_i = 1$ $0 \leq \alpha_i$, $\beta_i \leq 1$	$3n-2$

2 Production functions

2.1 Generalities

Producers' decisions can be derived from the maximization of profit (primal program) or from the minimization of production costs (dual problem). Both programs lead to the same results in terms of factor demands, of production, and thus of profit.

Primal program

In the primal program, the optimal production is determined by maximizing the producer's profit $\pi(Y, x_1 ..., x_n)$ under technology constraints:

$$\begin{cases} \max \pi(Y, x_1 ..., x_n) = PY - \sum_i w_i x_i \\ s.t. Y = f(x_1 ..., x_n) \end{cases}$$

Y is the quantity produced, P the market price of the product, and x_i and w_i respectively the quantity and price of factors used to produce.

Dual program

Here the producer minimizes his cost to produce a given level quantity Y.

$$\begin{cases} \min \sum_i w_i x_i \\ s.t. Y = f(x_1 ..., x_n) \end{cases}$$

The solution of this program is a set of factor demands $(x_1^*(w_1, ..., w_n, Y), ..., x_n^*(w_1, ..., w_n, Y))$ and, from this we can find the cost function $C(Y, w_1, ..., w_n) = \sum_i w_i x_i^*$ which gives the minimal cost to produce a given quantity Y.

Form, this we can compute the unit cost: $c = \frac{C}{Y}$, and the marginal cost: $Cm = \frac{\partial C}{\partial Y}$

The primal program described above can thus be rewritten as:

$$\max PY - C(Y, w_1, ..., w_n)$$

The first order condition of this program is:

$$\frac{\partial C}{\partial Y} = P$$

Namely, the marginal cost is equal to the market price of the product.

2.2 Leontief

The production technology is such that: $Y = \min\left\{\frac{x_i}{\alpha_i}\right\}$.

Factor demands

$$\begin{cases} \min \sum_i w_i x_i \\ s.t. Y = \min\left\{\frac{x_i}{\alpha_i}\right\} \end{cases}$$

This optimization program can also be written:

$$\begin{cases} \min \sum_i w_i x_i \\ s.t. g_i \leqslant 0, \forall i \end{cases}$$

With $g_i = Y - \frac{x_i}{\alpha_i}$.

$$\mathcal{L} = \sum_i w_i x_i + \sum_i \lambda_i \left(Y - \frac{x_i}{\alpha_i}\right)$$

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0 = w_i - \frac{\lambda_i}{\alpha_i} \Rightarrow \lambda_i = \alpha_i w_i$$

The complementarity slackness conditions are:

$$\min\{\lambda_i, g_i\} = 0, \forall i$$

Consequently:

$$w_i > 0 \Rightarrow g_i = 0, \forall i$$

$$\Rightarrow x_i = \alpha_i Y$$

Cost functions

$$C = \sum_i w_i x_i = Y \sum_i \alpha_i w_i$$

$$c = \frac{C}{Y} = \sum_i \alpha_i w_i$$

$$P = Cm = \sum_i \alpha_i w_i$$

2.3 Cobb-Douglas

The production technology is such that: $Y = A\prod_i x_i^{\alpha_i}$, with A and α_i paramters, and $\sum_i \alpha_i = 1$.

Factor demands

$$\begin{cases} \min \sum_i w_i x_i \\ s.t. Y = A\prod_i x_i^{\alpha_i} \end{cases}$$

$$\mathcal{L} = \sum_i w_i x_i - \lambda\left(A\prod_i x_i^{\alpha_i} - Y\right)$$

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0 = w_i - \lambda\left(A\frac{\alpha_i}{x_i}\prod_j x_j^{\alpha_j}\right) \Rightarrow \lambda = \frac{w_i x_i}{\alpha_i Y}, \forall i$$

$$x_j = x_i \frac{w_i \alpha_j}{w_j \alpha_i}, \forall i, j$$

$$Y = Ax_i \frac{w_i}{\alpha_i}\prod_j \left(\frac{\alpha_j}{w_j}^{\alpha_j}\right)$$

$$\Rightarrow x_i = \frac{Y\alpha_i}{Aw_i}\prod_j \left(\frac{w_j}{\alpha_j}\right)^{\alpha_j}$$

Cost functions

$$C = \sum_i w_i x_i = \frac{1}{A}Y\prod_j \left(\frac{w_j}{\alpha_j}\right)^{\alpha_j}$$

$$c = \frac{C}{Y} = \frac{1}{A}\prod_j \left(\frac{w_j}{\alpha_j}\right)^{\alpha_j}$$

$$P = Cm = \frac{\partial C}{\partial Y} = \frac{1}{A}\prod_j \left(\frac{w_j}{\alpha_j}\right)^{\alpha_j}$$

We can thus rewrite the demand function of production factors:

$$x_i = \frac{\alpha_i PY}{w_i}$$

Furthermore:

$$C = PY$$

and

$$c = Cm = P$$

2.4 CES

The production technology is such that: $Y = A\left(\sum_i \alpha_i x_i^{1-\frac{1}{\sigma}}\right)^{\frac{1}{1-\frac{1}{\sigma}}}$, with A and α_i parameters, and $\sum_i \alpha_i = 1$.

Factor demands

$$\begin{cases} \min \sum_i w_i x_i \\ s.t. Y = A\left(\sum_i \alpha_i x_i^{1-\frac{1}{\sigma}}\right)^{\frac{1}{1-\frac{1}{\sigma}}} \end{cases}$$

$$\mathcal{L} = \sum_i w_i x_i - \lambda\left(A\left(\sum_i \alpha_i x_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} - Y\right)$$

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0 = w_i - \lambda\left(A\alpha_i x_i^{-\frac{1}{\sigma}}\left(\sum_i \alpha_i x_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}}\right)$$

$$\Rightarrow \lambda = \frac{w_i}{A\alpha_i x_i^{-\frac{1}{\sigma}}\left(\sum_i \alpha_i x_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}}}, \forall i$$

$$x_j = x_i\left(\frac{w_i \alpha_j}{w_j \alpha_i}\right)^{\sigma}, \forall i, j$$

$$Y = A\left(\sum_i \alpha_i x_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$

$$\Rightarrow x_i = \frac{Y}{A}\left(\frac{\alpha_i}{w_i}\right)^{\sigma}\left(\sum_j \alpha_j^{\sigma} w_j^{1-\sigma}\right)^{-\frac{\sigma}{\sigma-1}}$$

Cost functions

$$C = \sum_i w_i x_i = \frac{Y}{A}\left(\sum_j \alpha_j^{\sigma} w_j^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$

$$c = \frac{C}{Y} = \frac{1}{A}\left(\sum_j \alpha_j^{\sigma} w_j^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$

$$P = Cm = \frac{1}{A}\left(\sum_j \alpha_j^{\sigma} w_j^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$

We can thus rewrite the demand function of production factors:

$$x_i = \alpha_i^{\sigma} A^{\sigma-1}\left(\frac{P}{w_i}\right)^{\sigma} Y$$

Furthermore:

$$C = PY$$

and

$$c = Cm = P$$

2.5 CET

The production technology is such that: $Y = A\left(\sum_i \alpha_i x_i^{\frac{\sigma+1}{\sigma}}\right)^{\frac{\sigma}{\sigma+1}}$, with A and α_i parameters, and $\sum_i \alpha_i = 1$.

Factor demands

$$\begin{cases} \min \sum_i w_i x_i \\ s.t. Y = A\left(\sum_i \alpha_i x_i^{\frac{\sigma+1}{\sigma}}\right)^{\frac{\sigma}{\sigma+1}} \end{cases}$$

$$\mathcal{L} = \sum_i w_i x_i - \lambda \left(A \left(\sum_i \alpha_i x_i^{\frac{\sigma+1}{\sigma}} \right)^{\frac{\sigma}{\sigma+1}} - Y \right)$$

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0 = w_i - \lambda \left(A \alpha_i x_i^{\frac{1}{\sigma}} \left(\sum_j \alpha_j x_j^{\frac{\sigma+1}{\sigma}} \right)^{-\frac{1}{\sigma+1}} \right)$$

$$\Rightarrow \lambda = \frac{w_i}{A \alpha_i x_i^{\frac{1}{\sigma}} \left(\sum_i \alpha_i x_i^{\frac{\sigma+1}{\sigma}} \right)^{-\frac{1}{\sigma+1}}}, \forall i$$

$$x_j = x_i \left(\frac{w_j \alpha_i}{w_i \alpha_j} \right)^{\sigma}, \forall i, j$$

$$Y = A \left(\sum_i \alpha_i x_i^{\frac{\sigma+1}{\sigma}} \right)^{\frac{\sigma}{\sigma+1}}$$

$$\Rightarrow x_i = \frac{Y}{A} \left(\frac{w_i}{\alpha_i} \right)^{\sigma} \left(\sum_j \alpha_j^{-\sigma} w_j^{1+\sigma} \right)^{-\frac{\sigma}{\sigma+1}}$$

Cost functions

$$C = \sum_i w_i x_i = \frac{Y}{A} \left(\sum_j \alpha_j^{-\sigma} w_j^{1+\sigma} \right)^{\frac{1}{\sigma+1}}$$

$$c = \frac{C}{Y} = \frac{1}{A} \left(\sum_j \alpha_j^{-\sigma} w_j^{1+\sigma} \right)^{\frac{1}{\sigma+1}}$$

$$P = Cm = \frac{1}{A}\left(\sum_j \alpha_j^{-\sigma} w_j^{1+\sigma}\right)^{\frac{1}{\sigma+1}}$$

We can thus rewrite the demand function of production factors:

$$x_i = \alpha_i^{-\sigma} A^{-\sigma-1}\left(\frac{w_i}{P}\right)^{\sigma} Y$$

Furthermore:

$$C = PY$$

and

$$c = Cm = P$$