AGRODEP African Growth & Development Policy modeling consortium FACILITATED BY IFPRI

DISTANCE LEARNING

PEP

SOLUTIONS

I. PRODUCER THEORY

1. Fill the table below

$f(l,k)$	MP_l	MP_k	$RTS_{l,k}$
$l+2k$	1	2	1/2
$al+bk$	a	b	a/b
$50\cdot l\cdot k$	50k	50l	k/l
$l^{0.25}k^{0.75}$	$0.25l^{-0.75}k^{0.75}$	$0.75k^{-0.25}l^{0.25}$	k/3l
$Cl^a k^b$	$Cal^{a-1}k^b$	$Cbl^a k^{b-1}$	ak/bl
$(l+2)(k+1)$	k+1	l+2	(k+1)/(l+2)

2. a) $f(x_1,x_2)=x_1+2x_2 \Rightarrow f(tx_1,tx_2)=tx_1+t2x_2=t\,(x_1+2x_2)=t\,f(x_1,x_2)\Rightarrow$ **Constant**

b) $f(x_1,x_2)=0.2x_1x_2^2 \Rightarrow f(tx_1,tx_2)=0.2tx_1(tx_2)^2=t^3(0.2x_1x_2^2)=t^3 f(x_1,x_2)\Rightarrow$ **Increasing**

c) $f(x_1,x_2)=x_1^{1/4}x_2^{3/4} \Rightarrow f(tx_1,tx_2)=(tx_1)^{1/4}(tx_2)^{3/4}=t\,(x_1^{1/4}x_2^{3/4})=tf(x_1,x_2)\Rightarrow$ **Constant**

3- Assume that your production function is given by $f(x,y)=x^{1/2}y^{3/2}$.

a) Write the expression of the marginal product of x.

$$MP_x=\frac{\partial f(x,y)}{\partial x}=\frac{1}{2}x^{-1/2}y^{3/2}$$

b) Assume a little increase of x, y is constant. Does the marginal product of x increase, decrease or remain constant?

To verify whether the marginal product is increasing or decreasing, we must verify the sign of the second derivative:

$$MP_{xx}=\frac{\partial MP_x}{\partial x}=-\frac{1}{4}\frac{y^{3/2}}{x^{3/2}}\leq 0\Rightarrow$$ **Decreasing**

c) Do the same for y.

$$MP_y = \frac{\partial f(x,y)}{\partial y} = \frac{3}{2} x^{1/2} y^{1/2}$$

d) If y increases, what happens on the marginal product of y?

$$MP_{yy} = \frac{\partial MP_y}{\partial y} = \frac{3}{4} \frac{x^{1/2}}{y^{1/2}} \geq 0 \Rightarrow \text{Marginal product of } y \text{ is increasing.}$$

e) What is the RTS between y and x?

$$RTS = \frac{MP_x}{MP_y} = \frac{\frac{1}{2} x^{-1/2} y^{3/2}}{\frac{3}{2} x^{1/2} y^{1/2}} = \frac{\frac{1}{2} y}{\frac{3}{2} x} = \frac{y}{3x}$$

f) What can you say about returns to scale of this function?

$$f(tx,ty) = (tx)^{1/2} (ty)^{3/2} = t\left(x^{1/2} y^{3/2}\right) = t^2 f(x,y) \Rightarrow \textbf{Increasing}$$

4. Consider the cost function: $C(y) = 4y^2 + 16$

a) What is the average cost function?

$$AC(y) = \frac{C(y)}{y} = \frac{4y^2 + 16}{y} = 4y + \frac{16}{y}$$

b) What is the marginal cost function?

$$MC(y) = \frac{\partial C(y)}{\partial y} = 8y$$

c) What is the level of production that minimizes the average cost?

$$\frac{\partial AC(y)}{\partial y} = 4 - \frac{16}{y^2} = 0 \Rightarrow 4y^2 = 16 \Rightarrow y = 2$$

d) For what production level is the average cost equal to the marginal cost?

$$AC(y) = MC(y) \Rightarrow 4y + \frac{16}{y} = 8y \Rightarrow 4y^2 + 16 = 8y^2 \Rightarrow y = 2$$

5. A firm in Los Angeles uses only one input. Its production function is: $f(x) = 4\sqrt{x}$ where x represents the number of input units. The output is sold at 100$ per unit, and the unitary cost of input is 50$.

a) Write an equation where the firm's profit depends on the output quantity.

$$\pi(y) = R(y) - C(y) = 100y - 50\frac{y^2}{16} = 100y - \frac{25y^2}{8}$$

b) What is the quantity of output that maximizes profit?

$$\frac{\partial \pi(y)}{\partial y} = 100 - 2 \cdot \frac{25y}{8} = 100 - \frac{50y}{8} = 0 \Rightarrow y = 16$$

What quantity of intput maximizes the profit?

Since $f(x) = 4\sqrt{x}$, then the quantity of input that maximizes profit is $16 = 4\sqrt{x} \Rightarrow x = 16$

How much is the maximal profit?

The maximum profit is thus $\pi(y) = 100y - \frac{25y^2}{8} = 100(16) - \frac{25(16)^2}{8} = 800$

c) Suppose that a tax of 20$ is added on every unit of output while input price benefits from a subsidy of 10$. What are the profits now?

$$\pi(y) = 80y - \frac{40y^2}{16} = 80(16) - \frac{40(16)^2}{16} = 640$$

d) Suppose that instead of setting up this tax and this grant, firm's profit is taxed at 50%. Give the profit after tax depending on the input's quantity.

$$\pi(x) = 0.5(R(x) - C(x)) = 0.5(100 f(x) - 50x) = 0.5\left(400\sqrt{x} - 50x\right) = 200\sqrt{x} - 25x$$

What is the production that maximizes profit?

$$\frac{\partial \pi(x)}{\partial x} = 200 \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{x}} - 25 = 0 \quad \Rightarrow 25\sqrt{x} = 100 \quad \Rightarrow x = 16 \quad \Rightarrow y = 16$$

What is the profit after tax?

$$\pi(x) = 200\sqrt{x} - 25x = 200\sqrt{16} - 25 \cdot 16 = 400$$

6. Jean is a gardener. He thinks that the number of nice plants, *h*, depends on the quantity of light, *l*, and the quantity of water, *w*. Jean notices that plants need twice more light than water, and also that one more or one less unit does not have any effect. Thus, his production function is given by :

$$h = \min\{l, 2w\}$$

a) Assume that Jean uses 1 unit of light. What is the minimal quantity of water he needs to produce a plant?

$h = \min\{l, 2w\} \Rightarrow l = h \quad and \quad w = \frac{h}{2}$ If $h=1$, then $l=1$ and $w=1/2$

b) Suppose that Jean wants to produce 4 plants. What are the minimal quantities of the two inputs he needs?

If $h=4$, then $l=4$ and $w=2$

c) What is the conditional demand function of light? What is the conditional demand function of water?

The conditional demand function of light depends on the prices of the two inputs (light and water) and the given level of production.

d) If *a* is the cost of one light unit, and *b* the cost of a unit of water. What is Jean's cost function?

$C = al + bw$

7. Flora is Jean's sister. She thinks that to get nice plants, you have to talk to them and give them fertilizer. Let *f* be the number of fertilizer's bag, and t the number of hours spent talking to plants.

The number of plants is exactly equal to $h = t + 2f$.

Assume that one fertilizer's bag costs a and a chat hour costs b.

a) If Flora does not use fertilizers, how many hours will she have to talk to plants?

$h = t + 2f, \quad if \quad f = 0 \Rightarrow h = t$

If Flora does not use fertilizer, she will have to talk 1 hour per plant.

If she does not talk at all, how many fertilizers' bags will she need for each plant?

$h = t + 2f, \quad if \quad t = 0 \Rightarrow h = 2f$

If she uses fertilizer and does not talk, she will use ½ bag per plant.

b) Suppose that $b<a/2$. In that case, if Flora wants to produce one plant, what will be cheaper for her: putting fertilizers or chatting?

If she only talks, then $h = t + 2f, \quad if \quad f = 0 \Rightarrow h = t$ and the cost per plant would be b.

If she only uses fertilizers, then $h = t + 2f, \quad if \quad t = 0 \Rightarrow h = 2f$ and the costs per plant would be $a/2$.

Since $b<a/2$ she will only use chatting.

c) What is Flora's cost function?

$$C(a,b,h) = \min\left\{b + \frac{a}{2}\right\} y$$

II. CONSUMER THEORY

1. Fill the table below

$U\ (x,y)$	MU_x	MU_y	MRS_{xy}
$2x+3y$	2	3	$\frac{2}{3}$
$2\sqrt{x}+y$	$\frac{1}{\sqrt{x}}$	1	$\frac{1}{\sqrt{x}}$
$x \cdot y$	y	x	$\frac{y}{x}$
$x^a y^b$	$ax^{a-1}y^b$	$bx^a y^{b-1}$	$\frac{ay}{bx}$
$x^a + y^b$	ax^{a-1}	by^{b-1}	$\frac{ay^{1-b}}{bx^{1-a}}$

2. Let the utility function of a given consumer be:

$$U(X,Y) = 2X(Y+1)$$

a. Derive the demand functions for each commodity.

$$L = 2X(Y+1) + \lambda(R - P_x X - P_y Y)$$

$$(1)\quad \frac{\partial L}{\partial X} = 2Y + 2 - \lambda P_x = 0$$

$$(2)\quad \frac{\partial L}{\partial Y} = 2X - \lambda P_y = 0$$

$$(3)\quad \frac{\partial L}{\partial \lambda} = R - P_x X - P_y Y = 0$$

$$\text{From (1) and (2)} \Rightarrow \lambda = \frac{2Y+2}{P_x} \text{ and } \lambda = \frac{2X}{P_y} \Rightarrow 2(Y+1)P_y = 2XP_x \Rightarrow\ X = \frac{(Y+1)P_y}{P_x}$$

$$\text{In (3)} \Rightarrow R - P_y Y - P_y - P_y Y = 0 \Rightarrow R - P_y = 2P_y Y$$

$$Y = \frac{R - P_y}{2P_y} \quad \text{and} \quad X = \frac{R + P_y}{2P_x}$$

b. Assume that $P_x = P_y = 1$. Determine the income R that allows reaching a utility of 32. Determine also corresponding consumptions.

$$X = \frac{R+1}{2} \text{ and } Y = \frac{R-1}{2}$$

$$U = 2X(Y+1)$$

$$32 = 2\left(\frac{R+1}{2}\right)\left(\frac{R-1}{2}+1\right)$$

$$32 = \frac{(R+1)^2}{2}$$

$$R = 7; \quad Y = 3 \quad and \quad X = 4$$

c. Now suppose that $P_x=2$ and $P_y=1$. Determine the new values of utility and consumptions when the income is the same as you found above.

$$U = 16; \quad Y = 3 \quad and \quad X = 2$$

3. Derive the demand functions for:

a. A Cobb-Douglas utility function: $U(x,y) = x^{\alpha} y^{\beta}$

$$x = \frac{\alpha \cdot R}{(\alpha+\beta)\cdot P_x} \quad and \quad y = \frac{\beta \cdot R}{(\alpha+\beta)\cdot P_y}$$

b. A Leontief utility function $U(x,y) = \min(\alpha x, \beta y)$

$$x = \frac{\beta \cdot R}{\beta \cdot P_x + \alpha \cdot P_y} \quad and \quad y = \frac{\alpha \cdot R}{\beta \cdot P_x + \alpha \cdot P_y}$$

c. A CES utility function $U(x,y) = A\left[\delta x^{-\rho} + (1-\delta) y^{-\rho}\right]^{\frac{-1}{\rho}}$

$$x = \frac{R}{P_x + P_y\left[\frac{(1-\delta)P_x}{\delta P_y}\right]^{\frac{1}{1+\rho}}} \quad and \quad y = \frac{R}{P_y + P_x\left[\frac{\delta P_y}{(1-\delta)P_x}\right]^{\frac{1}{1+\rho}}}$$

III. COMPETITIVE GENERAL EQUILIBRIUM MODEL

Assume that there are two (2) commodities produced in an economy. The production of each commodity i follows a Cobb-Douglas:

1. $XS_i = A_i LD_i^{\alpha_i} KD_i^{1-\alpha_i}$

Where XS_i, LD_i , KD_i represent output, labor demand and capital demand respectively, and A_i, α_i are parameters.

A. Assuming that the corresponding prices of output, labor and capital are P_i, w, r derive the demand function for labor and capital.

2. $LD_i = \frac{\alpha_i P_i XS_i}{w}$

3. $KD_i = \frac{(1-\alpha_i) P_i XS_i}{r}$

B. Let the utility function of the representative household also be a Cobb-Douglas function:

4. $U = B\, C_1^{\beta} C_2^{1-\beta}$

Where U, C_i represent utility and consumption of commodity i, and B, β are parameters. Assume further that household's income, Y, is simply the sum of return to labor and capital:

5. $Y = \sum_i \left(w\ LD_i + r\ KD_i \right)$

Assuming that consumer pays P_i for each commodity, derive the consumer demand functions:

6. $C_1 = \frac{\beta \cdot Y}{P_1}$

7. $C_2 = \frac{(1-\beta) \cdot Y}{P_2}$

C. Finally, assume that all markets are in equilibrium. Total labor supply, LS, is exogenous and the same goes for capital supply, KS. Derive equilibrium equation for each market:

8. $XS_i = C_i$

9. $\overline{LS} = \sum_i LD_i$

10. $\overline{KS} = \sum_i KD_i$

D. How many variables are there in the simple general equilibrium model you just derived?

There are 16 variables in the model: $XS_i, LD_i, KD_i, C_i, P_i, \overline{LS}, \overline{KS}, w, r, U, Y$.

(Variables indexed in i represent 2 variables as there are two commodities)

How many are endogenous and how many are exogenous?

There are 2 exogenous variables in the model, $\overline{LS}, \overline{KS}$, and 14 endogenous variables.

Is your model square (that is, do you get the same number of equations and endogenous variables)?

Yes, there are 14 endogenous variables and there are 14 equations (equations 1, 2, 3 and 8 represent two equations each as they are indexed in i)

E. What does the Walras' law imply?

The Walras' law says that if *n-1* markets are in equilibrium, then the last market is automatically in equilibrium as well. In our set of equations, one is thus redundant. We could define equation 8 over one commodity only and the other one would necessarily be equilibrated as well, following Walras' law).

Taking out one equation implies that our model would not be square anymore (we would have 14 endogenous variables but 13 equations). As only relative prices matter in a CGE model, the price of a commodity (or of a factor of production) is set to be the *numéraire* (therefore, fixed) and all other prices would then be expressed in terms of that price. Hence, our model would be square again.

IV. SOCIAL ACCOUNTING MATRIX

Receipts → / Expenses ↓	Factors of production		Agents			Productive activities			Acc.	TOTAL
	1.	2.	3.	4.	5.	6.	7.	8.	9.	(1 to 9)
1. Labor						WL_A	WL_I	WL_S		WL
2. Capital						RK_A	RK_I	RK_S		RK
3. Labor endowed household	WL									YH_{HW}
4. Capital endowed household		RK_{HC}			DIV					YH_{HW}
5. Firms		RK_F								YH_{HW}
6. Agriculture			$E_{A,HW}$	$E_{A,HC}$		$CI_{A,A}$	$CI_{A,I}$	$CI_{A,S}$	IV_A	VX_A
7. Industry			$E_{I,HW}$	$E_{I,HC}$		$CI_{I,A}$	$CI_{I,I}$	$CI_{I,S}$	IV_I	VX_A
8. Services			$E_{S,HW}$	$E_{S,HC}$		$CI_{S,A}$	$CI_{S,I}$	$CI_{S,S}$		VX_A
9. Accumulation			SH_{HW}	SH_{HC}	SF					ST
TOTAL	WL	RK	YH_{HW}	YH_{HC}	YF	VX_A	VX_I	VX_S	IT	

V. PARAMETRIZATION AND INTRODUCTION TO GAMS

Assuming that all prices are equal to one, what are the parameters value that would be both consistent with the SAM and the mathematical specification?

- *From equation 1, we can calibrate parameter* $v_j = {VA_j}/{XS_j}$

$$v_{AGR} = \frac{5{,}760 + 1{,}440}{9{,}000} = 0.80 \quad v_{IND} = \frac{7{,}560 + 11{,}340}{54{,}400} = 0.35$$

$$v_{SER} = \frac{15{,}540 + 5{,}720}{30{,}700} = 0.69$$

- *From equation 2, we can calibrate parameter* $io_j = {CI_j}/{XS_j}$

$$io_{AGR} = \frac{120 + 1{,}544 + 136}{9{,}000} = 0.20 \quad io_{IND} = \frac{2{,}526.9 + 21{,}709.1 + 11{,}264}{54{,}400} = 0.65$$

$$io_{SER} = \frac{275.5 + 5{,}815.5 + 3{,}349}{30{,}700} = 0.31$$

- *From equation 5, we can calibrate parameter* $aij_{i,j} = {DI_{i,j}}/{CI_j}$

$$aij_{AGR,AGR} = \frac{120}{120 + 1{,}544 + 136} = 0.067 \quad aij_{IND,AGR} = \frac{1{,}544}{120 + 1{,}544 + 136} = 0.858$$

$$aij_{SER,AGR} = \frac{136}{120 + 1{,}544 + 136} = 0.076$$

$$aij_{AGR,IND} = \frac{2{,}526.9}{2{,}526.9 + 21{,}709.1 + 11{,}264} = 0.071 \quad aij_{IND,IND} = \frac{21{,}709.1}{2{,}526.9 + 21{,}709.1 + 11{,}264} = 0.612$$

$$aij_{SER,IND} = \frac{11{,}264}{2{,}526.9 + 21{,}709.1 + 11{,}264} = 0.317$$

$$aij_{AGR,SER} = \frac{275.5}{275.5 + 5{,}815.5 + 3{,}349} = 0.029 \quad aij_{IND,SER} = \frac{5{,}815.5}{275.5 + 5{,}815.5 + 3{,}349} = 0.616$$

$$aij_{SER,SER} = \frac{3{,}349}{275.5 + 5{,}815.5 + 3{,}349} = 0.355$$

- *From equation 4, we can calibrate parameter* $\alpha_j = \frac{W \cdot LD_j}{PVA_j \cdot VA_j}$

$$\alpha_{AGR} = \frac{5{,}760}{5{,}760 + 1{,}440} = 0.80 \quad \alpha_{IND} = \frac{7{,}560}{7{,}560 + 11{,}340} = 0.4$$

$$\alpha_{SER} = \frac{15{,}540}{15{,}540 + 5{,}540} = 0.731$$

- *From equation 3, we can calibrate parameter* $A_j = \dfrac{VA_j}{LD_j^{\alpha_j} \cdot KD_j^{1-\alpha_j}}$

$$A_{AGR} = \frac{5,760 + 1,440}{5,760^{0.8} 1,440^{(1-0.8)}} = 1.649 \quad A_{IND} = \frac{7,560 + 11,340}{7,560^{0.4} 11,340^{(1-0.4)}} = 1.960$$

$$A_{SER} = \frac{15,540 + 5,540}{15,540^{0.731} 5,540^{(1-0.731)}} = 1.790$$

- *From equation 11, we can calibrate parameter* $\gamma_{i,h} = \dfrac{P_i \cdot C_{i,h}}{YH_h}$

$$\gamma_{AGR,HW} = \frac{4,329}{28,860} = 0.15 \quad \gamma_{IND,HW} = \frac{11,544}{28,860} = 0.4 \quad \gamma_{SER,HW} = \frac{10,101}{28,860} = 0.35$$

$$\gamma_{AGR,HC} = \frac{650}{13,000} = 0.05 \quad \gamma_{IND,HC} = \frac{3,900}{13,000} = 0.3 \quad \gamma_{SER,HC} = \frac{5,850}{13,000} = 0.45$$

- *From equation 7, we can calibrate parameter* $\lambda = \dfrac{YH_{hc} - DIV}{\sum_j R_j \cdot KD_j}$

$$\lambda = \frac{13,000 - 1,900}{18,500} = 0.6$$

- *From equation 12, we can calibrate parameter* $\mu_i = \dfrac{P_i \cdot INV_i}{IT}$

$$\mu_{AGR} = \frac{1,098.6}{10,986} = 0.1 \quad \mu_{IND} = \frac{9,887.4}{10,986} = 0.9$$

- *From equation 8, we can calibrate parameter* $\psi_h = \dfrac{SH_h}{YH_h}$

$$\psi_{HW} = \frac{2,886}{28,860} = 0.1 \quad \psi_{HC} = \frac{2,600}{13,000} = 0.2$$